Robust Dimension Reduction, Fusion Frames, and Grassmannian Packings

Gitta Kutyniok [*,1], Ali Pezeshki [2], Robert Calderbank [2], Taotao Liu [3]

Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544-1000, USA

Abstract

We consider estimating a random vector from its noisy projections onto low-dimensional subspaces constituting a *fusion frame*. A fusion frame is a collection of subspaces, for which the sum of the projection operators onto the subspaces is bounded below and above by constant multiples of the identity operator. We first determine the minimum mean-squared error (MSE) in linearly estimating the random vector of interest from its fusion frame projections, in the presence of white noise. We show that MSE assumes its minimum value when the fusion frame is *tight*. We then analyze the robustness of the constructed linear minimum MSE (LMMSE) estimator to erasures of the fusion frame subspaces. We prove that tight fusion frames consisting of equi-dimensional subspaces have maximum robustness (in the MSE sense) with respect to erasures of one subspace, and that the optimal subspace dimension depends on signal-to-noise ratio (SNR). We also prove that tight fusion frames consisting of equi-dimensional subspaces with equal pairwise chordal distances are most robust with respect to two and more subspace erasures. We call such fusion frames *equi-distance tight fusion frames,* and prove that the chordal distance between subspaces in such fusion frames meets the so-called *simplex bound,* and thereby establish connections between equi-distance tight fusion frames and *optimal Grassmannian packings.* Finally, we present several examples for construction of equi-distance tight fusion frames.

Key words: Distributed processing, Fusion frames, Grassmannian packings, LMMSE estimation, Packet encoding, Parallel processing, Robust dimension reduction, Subspace erasures.
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14. ABSTRACT

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1   Introduction

The notion of a *fusion frame* (or *frame of subspaces*) was introduced by Casazza and Kutyniok in [1] and further developed by Casazza et al. in [2]. A fusion frame for $\mathbb{R}^M$ is a finite collection of subspaces $\{\mathcal{W}_i\}_{i=1}^N$ in $\mathbb{R}^M$ such that there exist constants $0 < A \le B < \infty$ satisfying

$$A\|\mathbf{x}\|^2 \le \sum_{i=1}^N \|\mathbf{P}_i\mathbf{x}\|^2 \le B\|\mathbf{x}\|^2, \quad \text{for any } \mathbf{x} \in \mathbb{R}^M,$$

where $\mathbf{P}_i$ is the orthogonal projection onto $\mathcal{W}_i$. Alternatively, $\{\mathcal{W}_i\}_{i=1}^N$ is a fusion frame if and only if

$$A\,\mathbf{I} \le \sum_{i=1}^N \mathbf{P}_i \le B\,\mathbf{I}. \tag{1}$$

The constants A and B are called *(fusion) frame bounds*. An important class of fusion frames are *tight fusion frames*, for which A and B can be chosen to be equal and $\sum_{i=1}^N \mathbf{P}_i = A\,\mathbf{I}$. We note that the definition given in [1] and [2] for fusion frames apply to closed and weighted subspaces in any Hilbert space. However, since the scope of this paper is limited to non-weighted subspaces in $\mathbb{R}^M$, the definition of a fusion frame is only presented for this case.

A fusion frame can be viewed as a frame-like collection of low-dimensional subspaces. In frame theory, an input signal is represented by a collection of *scalars*, which measure the magnitudes of the projections of the signal onto frame vectors, whereas in fusion frame theory an input signal is represented by a collection of *vectors*, which are the projections of the signal onto the fusion frame subspaces. Similar to frames, fusion frames can be used to provide a redundant and non-unique representation of a signal. In fact, in many applications, where data has to be processed in a distributed manner by combining several locally processed data vectors, fusion frames can provide a more natural mathematical framework than frames. A few examples of such applications are as follows.

Distributed sensing: In distributed sensing, typically a large number of inexpensive sensors are deployed in an area to measure a physical quantity such

* Corresponding author

  Email addresses: kutyniok@math.princeton.edu (Gitta Kutyniok), pezeshki@math.princeton.edu (Ali Pezeshki), calderbk@math.princeton.edu (Robert Calderbank), taotaol@math.princeton.edu (Taotao Liu).

as temperature, sound, vibration, pressure, etc., or to keep an area under surveillance for target detection and tracking. Due to practical and economical factors, such as low communication bandwidth, limited signal processing power, limited battery life, or the topology of the surveillance area, the sensors are typically deployed in clusters, where each cluster includes a unit with higher computational and transmission power for local data processing. Thus, a typical large sensor network can be viewed as a redundant collection of sub-networks forming a set of subspaces. The gathered subspace information is submitted to a central processing station for joint processing. Some references that consider fusion frames for distributed sensing are [3],[4], and [5].

Parallel Processing: If a frame system is simply too large to handle effectively (from the numerical stand point), we can divide it into multiple small sub-systems for simpler and parallel processing. By introducing redundancy, when splitting the large system, we can introduce robustness against errors due to failure of a subsystem. Fusion frames provide a natural framework for splitting a large frame system into smaller subsystems and then recombining the subsystems.

Packet encoding: In digital media transmission, information bearing source symbols are typically encoded into a number of packets and then transmitted over a communication network, e.g., the internet. The transmitted packet may be corrupted during the transmission or completely lost due to, for example, buffer overflows. By introducing redundancy in encoding the symbols, according to an error correcting scheme, we can increase the reliability of the communication scheme. Fusion frames, as redundant collections of subspaces, can be used to produce a redundant representation of a source symbol. In the simplest form, we can think of each low-dimensional projection as a packet that carries some new information about the symbol. At the destination the packets can be decoded jointly to recover the transmitted symbol. The use of fusion frames for packet encoding is considered in [6].

In this paper, we consider estimating a *random vector* from its noisy projections onto low-dimensional subspaces constituting a fusion frame. As far as we know, optimal reconstruction of random vector signals from fusion frame measurements (or even frame measurements) has not been considered before, despite the fact that random vectors provide a natural way of modeling signals in many applications, including distributed sensing.

The optimal reconstruction of a random signal is different from the canonical reconstruction of a deterministic signal in a fusion frame that is considered in [2]. The canonical reconstruction strategy of a deterministic signal $\mathbf{x} \in \mathbb{R}^M$ from its fusion frame measurements involves using the fusion frame operator $\mathbf{S}\mathbf{x} = \sum_{i=1}^{N} \mathbf{P}_i\mathbf{x}$, which is invertible and self-adjoint. The deterministic signal $\mathbf{x}$ can then be recovered from its measurements $\{\mathbf{P}_i\mathbf{x}\}_{i=1}^{N}$ as $\mathbf{x} = \mathbf{S}^{-1}\mathbf{S}\mathbf{x} =$

$\sum_{i=1}^{N} \mathbf{S}^{-1}\mathbf{P}_i\mathbf{x}$. However, this strategy is not optimal in the MSE sense if $\mathbf{x}$ is a random vector.

When $\mathbf{x}$ is random, the (linear) strategy that achieves minimum MSE is *linear minimum mean-squared error (LMMSE) estimation* or *Wiener filtering*, which is well-known in the statistical signal processing literature (cf. [7, Ch.8]). We use this strategy to estimate the random vector $\mathbf{x}$ – assuming that its covariance matrix $\mathbf{R}_{xx} = E[\mathbf{x}\mathbf{x}^T]$ is nonsingular and known (but otherwise arbitrary) – from its noisy fusion frame projections. We determine the MSE in the LMMSE estimation of $\mathbf{x}$ and show that the MSE assumes its minimum value when the fusion frame is *tight*. Our analysis also clarifies the effect of additive white noise on signal estimation in fusion frames, which has not been studied before.

We then analyze the robustness of LMMSE estimators to erasures of the fusion frame subspaces. Erasures of subspaces can occur due to many factors in practice. In the distributed sensing example, a subspace erasure can occur due to a faulty or out of battery cluster of sensors, or due to loss of data during the transmission of local subspace information to the central processor. In scenarios where one or more sensor clusters are believed to be out of range for measuring the signal, or blocked by obstacles, their corresponding subspaces can be discarded on purpose. In the parallel processing example, an erasure can occur when a local processor crashes. In the packet encoding example, an erasure can occur when buffers in the network overflow.

Constructing frames that allow for robust reconstruction of a *deterministic signal* in the presence of frame element erasures has been considered by a number of authors. In [8], Goyal et al. show that a normalized frame is optimally robust against noise and one erasure (erasure of one element of the frame) if the frame is tight. Some ideas concerning multiple erasures were also presented. The work of Casazza and Kovačević [9] focuses mainly on designing frames, which maintain completeness under a particular number of erasures. Holmes and Paulsen [10] and Bodmann and Paulsen [11] study the robustness of frames under multiple erasures and show that maximal robustness with respect to the worst-case (maximum) Euclidean reconstruction error is achieved when the frame elements are equi-angular. The connection between equi-angular frames and equi-angular lines has also been explored by Strohmer and Heath in [12], where the so-called Grassmannian frames are introduced.

There are also a few papers that consider the construction of fusion frames for robust reconstruction of *deterministic signals* in the presence of subspace erasures. The main result in this context is due to Bodmann [6], who shows that a tight fusion frame is optimally robust against one subspace erasure if the dimensions of the subspaces are equal. He also proves that a tight fusion frame is optimally robust against multiple erasures if the subspaces satisfy the

so-called *equi-isoclinic* condition. The performance measure considered in [6] is the worst-case (maximum) Euclidean reconstruction error. The equi-isoclinic condition requires all pairs of subspaces to have the same set of principal angles. This condition is very restrictive and there are only a few known examples of fusion frames that satisfy this condition. The single erasure case discussed in [6] has also been studied by Casazza and Kutyniok in [13]. We emphasize that all the above work on robustness to erasures in frames and fusion frames deals with the case where the signal of interest is deterministic.

In this paper, we analyze the effect of subspace erasures on the performance of LMMSE estimators. We determine how the MSE of an LMMSE estimator, constructed based on the second-order statistics of the data in the absence of erasures, is affected by erasures. We limit our analysis to the case where the signal covariance matrix $\mathbf{R}_{xx}$ is of the form $\mathbf{R}_{xx} = \sigma_x^2 \mathbf{I}$. The case of a general $\mathbf{R}_{xx}$ is more involved and is outside the scope of this paper.

We prove that maximum robustness against one subspace erasure is achieved when the fusion frame is tight and all subspaces have equal dimensions where the optimal dimension depends on SNR. We also prove that a tight fusion frame consisting of equi-dimensional subspaces with equal pairwise *chordal distances* is maximally robust with respect to two and more subspace erasures. We call such fusion frames *equi-distance tight fusion frames*. We prove that the pairwise chordal distances between the subspaces in equi-distance tight fusion frames meet the so-called *simplex bound*, and thereby establish an intriguing connection between the construction of such fusion frames and *optimal Grassmannian packings* (cf. the excellent survey by Conway et al. [14]). This connection shows that optimal Grassmannian packings are fundamental for signal processing applications where low-dimensional projections are used for robust dimension reduction.

The paper is organized as follows. In Section 2, we derive the MSE in LMMSE estimation of a random vector from its noisy fusion frame projections. In Section 3, we analyze the robustness of LMMSE estimators to erasures of fusion frame subspaces and derive conditions for the construction of maximally robust fusion frames. Section 4 establishes a connection between equi-distance tight fusion frames and optimal Grassmannian packings. In Section 5, we give several examples for the construction of equi-distance tight fusion frames. Conclusions are drawn in Section 6.

2   LMMSE Estimation from Fusion Frame Measurements

Let $\mathbf{x} \in \mathbb{R}^M$ be a zero-mean random vector with covariance matrix $E[\mathbf{x}\mathbf{x}^T] = \mathbf{R}_{xx}$, which we wish to estimate from N noisy measurement vectors in a fusion

frame $\{\mathcal{W}_i\}_{i=1}^N$, with bounds $A \leq B$, in the presence of noise. In other words, we wish to estimate $\mathbf{x}$ from its noisy projections onto the subspaces $\{\mathcal{W}_i\}_{i=1}^N$. We take the dimension of the ith fusion frame subspace $\mathcal{W}_i$, $i = 1, 2, \ldots, N$ to be m_i.

Let $\mathbf{z}_i \in \mathbb{R}^M$, $i = 1, \ldots, N$ be the measurement vectors corresponding to $\{\mathcal{W}_i\}_{i=1}^N$. The measurement model for the ith subspace $\mathcal{W}_i$, $i = 1, \ldots, N$ is of the form

$$\mathbf{z}_i = \mathbf{P}_i \mathbf{x} + \mathbf{n}_i,$$

where $\mathbf{P}_i \in \mathbb{R}^{M \times M}$ is the orthogonal projection matrix onto the m_i-dimensional subspace $\mathcal{W}_i$, and $\mathbf{n}_i \in \mathbb{R}^M$ is the corresponding noise vector. Assume that the noise vectors for different subspaces are mutually uncorrelated, and that each noise vector is white with covariance matrix $E[\mathbf{n}_i \mathbf{n}_i^T] = \sigma_n^2 \mathbf{I}$, $i = 1, \ldots, N$. Further, assume that the signal vector $\mathbf{x}$ and the noise vectors $\mathbf{n}_i$, $i = 1, \ldots, N$ are uncorrelated.

Further, define the composite measurement vector $\mathbf{z} \in \mathbb{R}^{NM}$ and the composite projection matrix $\mathbf{P} \in \mathbb{R}^{NM \times M}$ as $\mathbf{z} = (\mathbf{z}_1^T \quad \mathbf{z}_2^T \quad \cdots \quad \mathbf{z}_N^T)^T$ and $\mathbf{P} = (\mathbf{P}_1^T \quad \mathbf{P}_2^T \quad \cdots \quad \mathbf{P}_N^T)^T$. Then, the composite covariance matrix between $\mathbf{x}$ and $\mathbf{z}$ can be written as

$$E\left[\begin{pmatrix} \mathbf{x} \\ \mathbf{z} \end{pmatrix} \begin{pmatrix} \mathbf{x}^T & \mathbf{z}^T \end{pmatrix} \right] = \begin{pmatrix} \mathbf{R}_{xx} & \mathbf{R}_{xz} \\ \mathbf{R}_{zx} & \mathbf{R}_{zz} \end{pmatrix} \in \mathbb{R}^{(N+1)M \times (N+1)M},$$

where

$$\mathbf{R}_{xz} = E[\mathbf{x}\mathbf{z}^T] = \mathbf{R}_{xx}\mathbf{P}^T = \mathbf{R}_{xx} \left(\mathbf{P}_1^T \quad \cdots \quad \mathbf{P}_N^T \right)$$

is the $M \times NM$ cross-covariance matrix between $\mathbf{x}$ and $\mathbf{z}$, $\mathbf{R}_{zx} = \mathbf{R}_{xz}^T$, and

$$\mathbf{R}_{zz} = E[\mathbf{z}\mathbf{z}^T] = \mathbf{P}\mathbf{R}_{xx}\mathbf{P}^T + \sigma_n^2 \mathbf{I} = \begin{pmatrix} \mathbf{P}_1 \\ \vdots \\ \mathbf{P}_N \end{pmatrix} \mathbf{R}_{xx} \left(\mathbf{P}_1^T \quad \cdots \quad \mathbf{P}_N^T \right) + \sigma_n^2 \mathbf{I} \qquad (2)$$

is the $NM \times NM$ composite measurement covariance matrix.

We wish to minimize the MSE in linearly estimating $\mathbf{x}$ from $\mathbf{z}$. The linear MSE minimizer is known to be the Wiener filter or the LMMSE filter $\mathbf{F} = \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}$, which estimates $\mathbf{x}$ by $\hat{\mathbf{x}} = \mathbf{F}\mathbf{z}$, e.g., see [7]. The error covariance matrix $\mathbf{R}_{ee}$ in

this estimation is given by

$$\begin{aligned}
\mathbf{R}_{ee} = E[\mathbf{e}\mathbf{e}^T] &= E\left[(\mathbf{x} - \mathbf{F}\mathbf{z})(\mathbf{x} - \mathbf{F}\mathbf{z})^T\right] \\
&= \mathbf{R}_{xx} - \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx} \\
&= \mathbf{R}_{xx} - \mathbf{R}_{xx}\mathbf{P}^T(\mathbf{P}\mathbf{R}_{xx}\mathbf{P}^T + \sigma_n^2\mathbf{I})^{-1}\mathbf{P}\mathbf{R}_{xx} \\
&= \left(\mathbf{R}_{xx}^{-1} + \frac{1}{\sigma_n^2}\mathbf{P}^T\mathbf{P}\right)^{-1},
\end{aligned}$$

where the last equality follows from the matrix inversion lemma (Sherman-Morrison-Woodbury formula) [15, p.50].

The MSE is obtained by taking the trace of $\mathbf{R}_{ee}$. Let ϕ_i, $i = 1, 2, \ldots, M$ be the ith eigenvalue of $\mathbf{R}_{xx}^{-1} + (1/\sigma_n^2)\mathbf{P}^T\mathbf{P}$ and assume $\phi_1 \geq \phi_2 \geq \cdots \geq \phi_M > 0$. Then, the MSE is

$$MSE = \text{tr}[\mathbf{R}_{ee}] = \sum_{i=1}^{M} \frac{1}{\phi_i}.$$

Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$ be the eigenvalues of $\mathbf{R}_{xx}$. Then, from (1), it follows that

$$\frac{1}{\lambda_i} + \frac{A}{\sigma_n^2} \leq \phi_i \leq \frac{1}{\lambda_i} + \frac{B}{\sigma_n^2}$$

or alternatively

$$\frac{1}{\frac{1}{\lambda_i} + \frac{B}{\sigma_n^2}} \leq \frac{1}{\phi_i} \leq \frac{1}{\frac{1}{\lambda_i} + \frac{A}{\sigma_n^2}}.$$

Therefore, we have the following lower and upper bounds for the MSE:

$$\sum_{i=1}^{M} \frac{1}{\frac{1}{\lambda_i} + \frac{B}{\sigma_n^2}} \leq \left(MSE = \sum_{i=1}^{M} \frac{1}{\phi_i}\right) \leq \sum_{i=1}^{M} \frac{1}{\frac{1}{\lambda_i} + \frac{A}{\sigma_n^2}}.$$

The lower bound is achieved when the fusion frame is tight. That is, when $A = B$ and

$$\sum_{\ell=1}^{N} \mathbf{P}_\ell = A\mathbf{I}. \tag{3}$$

Taking the trace from both sides of (3) yields the bound A as

$$A = \frac{1}{M} \sum_{\ell=1}^{N} m_\ell. \tag{4}$$

Thus, the MSE is given by

$$MSE = \sum_{i=1}^{M} \frac{\sigma_n^2 \lambda_i}{\sigma_n^2 + \frac{\lambda_i}{M} \sum_{\ell=1}^{N} m_\ell}. \tag{5}$$

Remark 2.1 When $\mathbf{R}_{xx} = \sigma_x^2 \mathbf{I}$, the MSE expression in (5) reduces to

$$MSE = \frac{M \sigma_n^2 \sigma_x^2}{\sigma_n^2 + \frac{\sigma_x^2}{M} \sum_{\ell=1}^{N} m_\ell}. \tag{6}$$

3 Robustness to Subspace Erasures

We now consider the case where subspace erasures occur, that is when measurement vectors from one or more subspaces are lost or discarded. We wish to determine the MSE when the LMMSE filter $\mathbf{F}$, which is calculated based on the full composite covariance matrix in (2), is applied to the composite measurement vector with erasures. We do not wish to recalculate the LMMSE filter every time an erasure occurs. Recalculating the LMMSE filter requires calculating the inverse of the composite covariance matrix of the remaining measurement vectors, which is intractable from a computational standpoint.

In this section, we show how the subspaces in the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ must be selected so that the MSE is minimized under subspace erasures. In our analysis we assume that $\{\mathcal{W}_i\}_{i=1}^N$ is tight with bound A given by (4). For the sake of simplicity, we limit our analysis to the case where the signal covariance matrix is $\mathbf{R}_{xx} = \sigma_x^2 \mathbf{I}$. The case where $\mathbf{R}_{xx}$ is a general positive definite matrix is more involved and is outside the scope of this paper.

Let $\mathbb{S} \subset \{1, 2, \ldots, N\}$ be the set of indices corresponding to the erased subspaces. Then, the composite measurement vector with erasures $\tilde{\mathbf{z}} \in \mathbb{R}^{NM}$ may be expressed as

$$\tilde{\mathbf{z}} = (\mathbf{I} - \mathbf{E})\mathbf{z},$$

where $\mathbf{E}$ is an $NM \times NM$ block-diagonal erasure matrix whose ith $M \times M$ diagonal block is a zero matrix, if $i \notin \mathbb{S}$, or an identity matrix, if $i \in \mathbb{S}$. In other words, in $\tilde{\mathbf{z}}$ the measurement vectors associated with the erased subspaces are set to zero.

The estimate of $\mathbf{x}$ is given by $\tilde{\mathbf{x}} = \mathbf{F}\tilde{\mathbf{z}}$, where $\mathbf{F} = \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}$ is the (no-erasure) LMMSE filter. The error covariance matrix $\widetilde{\mathbf{R}}_{ee}$ for this estimate is given by

$$\begin{aligned}
\widetilde{\mathbf{R}}_{ee} &= E\left[(\mathbf{x} - \tilde{\mathbf{x}})(\mathbf{x} - \tilde{\mathbf{x}})^T\right] \\
&= E\left[(\mathbf{x} - \mathbf{F}(\mathbf{I} - \mathbf{E})\mathbf{z})(\mathbf{x} - \mathbf{F}(\mathbf{I} - \mathbf{E})\mathbf{z})^T\right] \\
&= \mathbf{R}_{xx} - \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}(\mathbf{I} - \mathbf{E})\mathbf{R}_{zx} - \mathbf{R}_{xz}(\mathbf{I} - \mathbf{E})^T\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx} \\
&\quad + \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}(\mathbf{I} - \mathbf{E})\mathbf{R}_{zz}(\mathbf{I} - \mathbf{E})^T\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx}.
\end{aligned}$$

We can rewrite $\widetilde{\mathbf{R}}_{ee}$ as

$$\widetilde{\mathbf{R}}_{ee} = \mathbf{R}_{ee} + \overline{\mathbf{R}}_{ee},$$

where $\mathbf{R}_{ee} = \mathbf{R}_{xx} - \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx}$ is the no-erasure error covariance matrix, and

$$\overline{\mathbf{R}}_{ee} = \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}\mathbf{E}\mathbf{R}_{zz}\mathbf{E}^T\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx}$$

is the extra covariance matrix due to erasures. The MSE is given by

$$MSE = \text{tr}[\mathbf{R}_{ee}] = MSE_0 + \overline{MSE},$$

where $MSE_0 = \text{tr}[\mathbf{R}_{ee}]$ is the no-erasure MSE in (6) and

$$\begin{aligned}
\overline{MSE} &= \text{tr}[\overline{\mathbf{R}}_{ee}] \\
&= \text{tr}[\mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}\mathbf{E}\mathbf{R}_{zz}\mathbf{E}^T\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx}] \\
&= \text{tr}[\sigma_x^4\mathbf{P}^T(\sigma_x^2\mathbf{PP}^T + \sigma_n^2\mathbf{I})^{-1}\mathbf{E}(\sigma_x^2\mathbf{PP}^T + \sigma_n^2\mathbf{I})\mathbf{E}^T(\sigma_x^2\mathbf{PP}^T + \sigma_n^2\mathbf{I})^{-1}\mathbf{P}]
\end{aligned}$$

is the extra MSE due to erasures.

From the matrix inversion lemma [15, p.50], we have

$$\begin{aligned}
(\sigma_x^2\mathbf{PP}^T + \sigma_n^2\mathbf{I})^{-1} &= \frac{1}{\sigma_n^2}\mathbf{I} - \frac{1}{\sigma_n^4}\mathbf{P}(\frac{1}{\sigma_n^2}\mathbf{P}^T\mathbf{P} + \frac{1}{\sigma_x^2}\mathbf{I})^{-1}\mathbf{P}^T \\
&= \frac{1}{\sigma_n^2}\mathbf{I} - \frac{1}{\sigma_n^2}\frac{\sigma_x^2}{A\sigma_x^2 + \sigma_n^2}\mathbf{PP}^T,
\end{aligned} \quad (7)$$

where the second equality follows by using $\mathbf{P}^T\mathbf{P} = \sum_{i=1}^N \mathbf{P}_i = A\mathbf{I}$.

Using (7), we can simplify the expression for $\overline{MSE}$ to

$$\begin{aligned}
\overline{MSE} = \text{tr}[\overline{\mathbf{R}}_{ee}] &= \alpha^2\text{tr}[\mathbf{P}^T\mathbf{E}(\sigma_x^2\mathbf{PP}^T + \sigma_n^2\mathbf{I})\mathbf{E}\mathbf{P}] \\
&= \alpha^2\text{tr}\left[\sigma_x^2\left(\sum_{i\in\mathbb{S}}\mathbf{P}_i\right)^2 + \sigma_n^2\left(\sum_{i\in\mathbb{S}}\mathbf{P}_i\right)\right], \quad (8)
\end{aligned}$$

where $\alpha = \sigma_x^2/(\sigma_x^2 A + \sigma_n^2)$. The last equality in (8) follows by considering the action of the erasure matrix $\mathbf{E}$.

We now show how the subspaces in the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ must be constructed so that the total MSE is minimized for a given number of erasures. We consider three scenarios: one subspace erasure, two subspace erasures, and more than two subspace erasures.

3.1   One Subspace Erasure

If only one of the subspaces, say the ith subspace, is erased, then MSE is given by

$$
\begin{aligned}
MSE &= MSE_0 + \overline{MSE} \\
&= MSE_0 + \operatorname{tr}[\alpha^2(\sigma_x^2 + \sigma_n^2)\mathbf{P}_i] \\
&= \frac{M\sigma_x^2\sigma_n^2}{\sigma_n^2 + \frac{\sigma_x^2}{M}\sum\limits_{\ell=1}^{N} m_\ell} + \frac{\sigma_x^4(\sigma_x^2 + \sigma_n^2)}{\left(\sigma_n^2 + \frac{\sigma_x^2}{M}\sum\limits_{\ell=1}^{N} m_\ell\right)^2} m_i,
\end{aligned}
\tag{9}
$$

where $m_i = \operatorname{tr}[\mathbf{P}_i]$ is the dimension of the ith subspace $\mathcal{W}_i$.

The erasure can occur for any of the subspaces. Thus, we have to choose $m_i = m$ for all $i = 1, \ldots, N$ so that any one-erasure results in the same amount of performance degradation. This reduces the MSE expression (9) to

$$
MSE = \frac{M\sigma_x^2\sigma_n^2}{(Nm\sigma_x^2/M + \sigma_n^2)} + \frac{\sigma_x^4(\sigma_x^2 + \sigma_n^2)m}{(Nm\sigma_x^2/M + \sigma_n^2)^2}.
$$

As a function of m, $MSE = MSE(m)$ has a maximum at $m = \tilde{m}$, where

$$
\tilde{m} = \frac{M}{N} \frac{(N-1)\sigma_n^4 - \sigma_x^2\sigma_n^2}{((N+1)\sigma_n^2 + \sigma_x^2)(1 - 2\sigma_x^2)}.
$$

The MSE is monotonically increasing for $m < \tilde{m}$ and monotonically decreasing for $m > \tilde{m}$. The smallest value m can take under the constraint that the set of m-dimensional subspace $\{\mathcal{W}_i\}_{i=1}^{M}$ remains a tight fusion frame is $m_{min} = \lceil M/N \rceil$, where $\lceil \cdot \rceil$ denotes integer ceiling. We take the largest value m can take to be $m_{max} \leq M$. The maximum allowable dimension m_{max} is determined by practical considerations. In the distributed sensing problem it is the maximum number of sensors we can deploy in a cluster. In the parallel processing problem it is determined by the maximum computational load that the local processors can handle, and in the packet encoding problem it corresponds to the maximum amount of new information (minimum amount of redundancy) we can include in a packet, while achieving an error correction goal.

We have the following theorem.

Theorem 3.1 *The MSE due to the erasure of one subspace is minimized when*

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all subspaces in $\{\mathcal{W}_i\}_{i=1}^N$ have equal dimension $m = m^*$, where

$$
m^* = \begin{cases} m_{min}, & \text{if } m_{max} \leq \tilde{m} \text{ or} \\ & \text{if } m_{min} \leq \tilde{m} \leq m_{max} \text{ and } MSE(m_{min}) \leq MSE(m_{max}), \\ m_{max}, & \text{otherwise.} \end{cases}
$$

3.2   Two Subspace Erasures

When two subspaces, say the ith subspace and the jth subspace, are erased or discarded, the total MSE is given by

$$
\begin{aligned}
MSE &= MSE_0 + \overline{MSE} \\
&= MSE_0 + \alpha^2 \text{tr}[\sigma_x^2(\mathbf{P}_i + \mathbf{P}_j)^2 + \sigma_n^2(\mathbf{P}_i + \mathbf{P}_j)].
\end{aligned}
$$

We take the dimension of all subspaces to be equal to a given m in order to fix the performance against one subspace erasures. This fixes MSE_0 and reduces the minimization of MSE to minimizing the extra MSE, which is given by

$$
\overline{MSE} = 2\alpha^2(\sigma_x^2 + \sigma_n^2)m + 2\alpha^2 \sigma_x^2 \text{tr}[\mathbf{P}_i\mathbf{P}_j].
$$

To minimize $\overline{MSE}$ we have to choose $\mathcal{W}_i$ and $\mathcal{W}_j$, so that $\text{tr}[\mathbf{P}_i\mathbf{P}_j]$ is minimized. Since $\mathbf{P}_i$ and $\mathbf{P}_j$ are orthogonal projection matrices onto $\mathcal{W}_i$ and $\mathcal{W}_j$, the eigenvalues of $\mathbf{P}_i\mathbf{P}_j$ are squares of the cosines of the *principal angles $\theta_\ell(i, j)$*, $\ell = 1, \ldots, M$ between $\mathcal{W}_i$ and $\mathcal{W}_j$. Therefore,

$$
\text{tr}[\mathbf{P}_i\mathbf{P}_j] = \sum_{\ell=1}^M \cos^2 \theta_\ell(i, j) = M - d_c^2(i, j), \tag{10}
$$

where

$$
d_c(i, j) = \left(\sum_{\ell=1}^M \sin^2 \theta_\ell(i, j) \right)^{1/2}
$$

is known as the *chordal distance* [14] between $\mathcal{W}_i$ and $\mathcal{W}_j$.

Thus, we need to maximize the chordal distance $d_c(i, j)$. Since this has to be done for any two subspace erasures, i.e., for any pair (i, j), $i \neq j$, we have to construct the subspaces $\{\mathcal{W}_i\}_{i=1}^N$ so that any such pair has maximum chordal distance.

In Section 4, we will prove that the subspaces in a fusion frame consisting of equi-dimensional and equi-distance (equi-chordal distance) subspaces have maximal chordal distance if and only if the fusion frame is tight. We call such a fusion frame an *equi-distance tight fusion frame* and the subspaces

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corresponding to it *maximal equi-distance subspaces*. We note that maximal equi-distance does not mean that the principal angles between any pair of subspaces must be equal. Therefore, this is a more relaxed requirement than the equi-isoclinic condition in [6].

We have the following theorem.

Theorem 3.2 *The MSE due to two subspace erasures is minimized when the m-dimensional subspaces in the tight fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ are maximal equi-distance subspaces.*

We defer the construction of maximal equi-distance subspaces to Section 4, where we explain the connection between this construction and the problem of optimal packing of N planes in a Grassmannian space [16,14].

3.3   More Than Two Subspace Erasures

We now consider the case where more than two subspaces are erased or discarded. Let the subspaces $\{\mathcal{W}_i\}_{i=1}^N$ have equal dimension $m \geq \lceil M/N \rceil$ and equal pairwise chordal distance d_c, so as to fix the performance against one- and two-erasures. Then, $\overline{MSE}$ can be written as

$$
\begin{aligned}
\overline{MSE} &= \alpha^2 \mathrm{tr}\left[\sigma_x^2 \left(\sum_{i \in \mathbb{S}} \mathbf{P}_i\right)^2 + \sigma_n^2 \left(\sum_{i \in \mathbb{S}} \mathbf{P}_i\right)\right] \\
&= \alpha^2(\sigma_x^2 + \sigma_n^2)\sum_{i \in \mathbb{S}} \mathrm{tr}[\mathbf{P}_i] + \alpha^2\sigma_x^2 \sum_{i \in \mathbb{S}}\sum_{j \in \mathbb{S}, j \neq i} \mathrm{tr}[\mathbf{P}_i\mathbf{P}_j] \\
&= \alpha^2(\sigma_x^2 + \sigma_n^2)|\mathbb{S}|m + \alpha^2\sigma_x^2|\mathbb{S}|(|\mathbb{S}| - 1)(M - d_c^2).
\end{aligned}
$$

Similar to the two-erasure case, $\overline{MSE}$ is minimized when d_c^2 takes its maximum value. Thus, we have the following theorem.

Theorem 3.3 *The MSE due to k erasures, $3 \leq k < N$ is minimized when the m-dimensional subspaces in the tight fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ are maximal equi-distance subspaces.*

4   Connections between Tight Fusion Frames and Optimal Packings

In this section, we show that tight fusion frames that consist of equi-dimensional and equi-distance subspaces are closely related to optimal packings of

subspaces. We start by reviewing the classical packing problem for subspaces [16,14].

Classical Packing Problem. For given m, M, N, find a set of m-dimensional subspaces $\{\mathcal{W}_i\}_{i=1}^{N}$ in $\mathbb{R}^M$ such that $\min_{i \neq j} d_c(i, j)$ is as large as possible. In this case we call $\{\mathcal{W}_i\}_{i=1}^{N}$ an *optimal packing*.

This problem was reformulated by Conway et al. in [14] by describing m-dimensional subspaces in $\mathbb{R}^M$ as points on a sphere of radius $\frac{1}{2}(M-1)(M+2)$. This usually provides a lower-dimensional representation than the Plücker embedding. This idea was then used to prove the optimality of many new packings by employing results from sphere packing theory such as Rankin bounds for spherical codes. In what follows, we briefly describe the embedding of the Grassmannian manifold $G(m, M)$ of m-dimensional subspaces of $\mathbb{R}^M$, as it was described in [14]. The basic idea is to identify an m-dimensional subspace $\mathcal{W}$ with the traceless part of the projection matrix $\mathbf{Q}_1$ associated with $\mathcal{W}$, i.e., with $\overline{\mathbf{Q}}_1 = \mathbf{Q}_1 - \frac{m}{M}\mathbf{I}$. This yields an isometric embedding of $G(m, M)$ into the sphere of radius $\sqrt{\frac{m(M-m)}{M}}$ in $\mathbb{R}^{\frac{1}{2}(M-1)(M+2)}$, where the distance measure is the chordal distance between two projections. The chordal distance $d_c(\mathbf{Q}_1, \mathbf{Q}_2)$ between two projection matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ is given by $d_c(\mathbf{Q}_1, \mathbf{Q}_2) = \frac{1}{\sqrt{2}}\|\mathbf{Q}_1 - \mathbf{Q}_2\|_2$, and is equal to $\frac{1}{\sqrt{2}}$ times the straight-line distance between the projection matrices. This is the reason that $d_c(\mathbf{Q}_1, \mathbf{Q}_2)$ is called *chordal* distance. Conway et al. [14] deduced from this particular embedding the following result.

Theorem 4.1 [14] *Each packing of m-dimensional subspaces $\{\mathcal{W}_i\}_{i=1}^{N}$ in $\mathbb{R}^M$ satisfies*

$$d_c^2(i, j) \leq \frac{m(M - m)}{M} \frac{N}{N - 1}, \quad i, j = 1, \ldots, N.$$

The upper bound is referred to as the *simplex bound*. The above theorem implies that if the pairwise chordal distances between a set of m-dimensional subspaces of $\mathbb{R}^M$ meet the simplex bound those subspaces form an optimal packing, as the minimum of chordal distances cannot grow any further.

We now establish a connection between tight fusion frames and optimal packings.

4.1 Equi-Dimensional Subspaces

Consider a tight fusion frame $\{\mathcal{W}_i\}_{i=1}^{N}$, with bound A, consisting of N m-dimensional subspaces that do not necessarily have equal pairwise chordal

distances. Since $\{\mathcal{W}_i\}_{i=1}^N$ is tight, we have

$$A\mathbf{I} = \sum_{i=1}^{N} \mathbf{P}_i. \tag{11}$$

On the one hand, we can apply the trace and employ the fact that $\text{tr}[\mathbf{P}_i] = m$ for each i, to obtain

$$AM = Nm. \tag{12}$$

On the other hand, we can multiply (11) from left by $\mathbf{P}_j$ to get

$$(A-1)\,\mathbf{P}_j = \sum_{i=1,i\neq j}^{N} \mathbf{P}_j\mathbf{P}_i, \quad j = 1,\ldots,N.$$

We can then take the trace, employ the fact that $\text{tr}[\mathbf{P}_j] = m$ for each j, and use (10), to obtain

$$(A-1)\,m = \sum_{i=1,i\neq j}^{N} \text{tr}[\mathbf{P}_j\mathbf{P}_i] = (N-1)m - \sum_{i=1,i\neq j}^{N} d_c^2(i,j). \tag{13}$$

Equations (12) and (13) together prove the following result concerning the value of the fusion frame bound.

Proposition 4.2 *A tight fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ with bound A and m-dimensional subspaces satisfies*

$$A = \frac{Nm}{M} = N - \sum_{i=1,i\neq j}^{N} \frac{d_c^2(i,j)}{m}, \quad j = 1,\ldots,N.$$

4.2   Equi-Dimensional and Equi-Distance Subspaces

We now turn our attention to tight fusion frames $\{\mathcal{W}_i\}_{i=1}^N$ consisting of equi-dimensional and equi-distance subspaces, where the common dimension is m and the common chordal distance is d_c. From Proposition 4.2, it follows that

$$\frac{Nm}{M} = N - (N-1)\frac{d_c^2}{m}.$$

Thus, d_c^2 is given by

$$d_c^2 = \frac{m(M-m)}{M}\frac{N}{N-1}, \tag{14}$$

which shows that d_c^2 precisely equals the simplex bound.

Next we will study whether this condition is sufficient. That is, we wish to know whether a fusion frame consisting of equi-dimensional subspaces whose pairwise chordal distances are equal to the simplex bound is necessarily tight.

Consider a fusion frame $\{\mathcal{W}_i\}_{i=1}^N$, consisting of N m-dimensional subspaces with pairwise chordal distances d_c equal to the simplex bound. Let $\pi_1, \ldots, \pi_M$ be the eigenvalues of $\mathbf{P}^T\mathbf{P} = \sum_{i=1}^N \mathbf{P}_i$. Since $\{\mathcal{W}_i\}_{i=1}^N$ is a fusion frame for $\mathbb{R}^M$, we have $\pi_\ell > 0$, $\ell = 1, 2, \ldots, M$, and the sum of π_ℓ's is given by

$$\sum_{\ell=1}^M \pi_\ell = \text{tr}[\mathbf{P}^T\mathbf{P}] = \sum_{i=1}^N \text{tr}[\mathbf{P}_i] = Nm. \tag{15}$$

The sum of π_i^2's can be written as

$$\sum_{\ell=1}^M \pi_\ell^2 = \text{tr}[\mathbf{P}^T\mathbf{P}\mathbf{P}^T\mathbf{P}]$$

$$= \sum_{i=1}^N \sum_{j=1}^N \text{tr}[\mathbf{P}_i\mathbf{P}_j]$$

$$= \sum_{i=1}^N \sum_{j=1, j\neq i}^N \text{tr}[\mathbf{P}_i\mathbf{P}_j] + \sum_{i=1}^N \text{tr}[\mathbf{P}_i]$$

$$= N(N-1)(m - d_c^2) + Nm,$$

where the last equality follows from (10). Inserting the value of the simplex bound, we obtain

$$\sum_{\ell=1}^M \pi_\ell^2 = \frac{m^2 N^2}{M}. \tag{16}$$

To conclude that (15) together with (16) implies tightness of the fusion frame, we consider the problem of minimizing the function $\sum_{\ell=1}^M \pi_\ell^2$ under the constraint that $\pi_1, \ldots, \pi_M$ is a sequence of nonnegative values which sum up to $\sum_{\ell=1}^M \pi_\ell = Nm$. Using the method of Lagrange multipliers, we see that the minimum is achieved when all π_ℓ's are equal to Nm/M. This implies that (15) and (16) can be simultaneously satisfied only when

$$\pi_1 = \cdots = \pi_M = \frac{Nm}{M}.$$

From this relation, it follows that $\{\mathcal{W}_i\}_{i=1}^N$ is a tight fusion frame. Therefore, we have the following theorem.

Theorem 4.3 *Let $\{\mathcal{W}_i\}_{i=1}^N$ be a fusion frame of m-dimensional subspaces with equal pairwise chordal distances d_c. Then the fusion frame is tight if and only if d_c equals the simplex bound.*

An immediate consequence of Theorem 4.3 is as follows.

Corollary 4.4 *Equi-distance tight fusion frames are optimal Grassmannian packings.*

5 Construction of Equi-Distance Tight Fusion Frames

In this section we present a few examples to illustrate the richness, but also the difficulty of constructing fusion frames with special properties such as tightness, equi-dimension, and equi-distance. The optimal packing of N planes in the Grassmannian space $G(m, M)$ is a difficult mathematical problem, the solution to which is known only for special values of N,m, and M. In fact, even optimal packing of lines ($m = 1$) or equivalently constructing equi-angular lines is a deep mathematical problem. The reader is referred to [12] for a review of problems which are equivalent to the construction of equi-angular lines. For the construction of optimal packings with higher-dimensional subspaces we refer the reader to [14,16,17]. We would also like to draw the reader's attention to N. J. Sloane's webpage [18], which includes many examples of Grassmannian packings.

Example 5.1 As our first example for construction of equi-distance tight fusion frames, we use a result obtained by Calderbank et al. [17] for construction of optimal packings. The procedure is as follows. Choose p to be a prime which is either 3 or congruent to -1 modulo 8. Then there exists an explicit construction which produces a tight fusion frame $\{\mathcal{W}_i\}_{i=1}^{p(p+1)/2}$ in $\mathbb{R}^p$ with

$$m_i = \frac{p-1}{2} \quad \text{and} \quad d_c^2 = \frac{(p+1)^2}{4(p+2)} \quad \text{for all } i, j = 1, \ldots, \frac{p(p+1)}{2},$$

where m_i denotes the dimension of the ith subspace. From Proposition 4.2 it follows that the bound of this fusion frame equals

$$A = \frac{p^2 - 1}{4}.$$

As a particular example of this construction we briefly outline the equi-distance tight fusion frame we obtain for $p = 7$. For this, let $Q = \{q_i\}_{i=1}^3 = \{1, 2, 4\}$ denote the nonzero quadratic residues modulo 7, and $R = \{3, 5, 6\}$ the nonresidues. Further, let $\mathbf{H}$ be a 4×4 Hadamard matrix, e.g.,

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Finally, we denote the coordinate vectors in $\mathbb{R}^7$ by $\mathbf{e}_i$, $0 \leq i \leq 6$, and set $C = \sqrt{2}$ and $k = 3$. Then we define 4 three-dimensional planes $\mathbf{L}_j$, $1 \leq j \leq 4$

to be spanned by the vectors

$$\mathbf{e}_{q_i} + C\mathbf{H}_{ij}\mathbf{e}_{kq_i}, \quad 1 \leq i \leq 3.$$

For each $\mathbf{L}_j$, we obtain 6 further planes by applying the cyclic permutation of coordinates $\mathbf{e}_i \mapsto \mathbf{e}_{(i+1) \mod 7}$. This yields 28 three-dimensional planes in $\mathbb{R}^7$, which form a tight fusion frame with bound 12. Moreover, the chordal distance between each pair of them equals $d_c^2 = \frac{16}{9}$.

This construction is based on employing properties of special groups, in this case the Clifford group. We remark that this is closely related with the construction of error-correcting codes.

Example 5.2 This example considers the construction of an equi-distance tight fusion frame for a dimension not covered by Example 5.1 by employing the theory of Eisenstein integers. More precisely, the subspaces will be generated by the minimal elements of a special lattice. For this, we let $\mathcal{E} = \{a + \omega b : a, b \in \mathbb{Z}\}$ denote the Eisenstein integers, where $\omega = \frac{-1+i\sqrt{3}}{2}$ is a complex root of unity. The three-dimensional complex lattice E_6^* over $\mathcal{E}$ is then defined by its generator matrix

$$\begin{pmatrix} \sqrt{-3} & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

It can be shown that the minimal norm of a non-zero element in E_6^* is $\frac{4}{3}$. Out of the set of minimal elements, we now select the following nine:

$$(1, -1, 0), \ (1, 0, -1), \ (0, 1, -1), \ (\omega, -1, 0),$$

$$(0, \omega, -1), \ (-1, 0, \omega), \ (\omega, 0, -1), \ (-1, \omega, 0), \ (0, -1, \omega).$$

Multiplied by the 6th roots of unity, this yield 9 planes in $\mathbb{C}^3$. Using the canonical mapping of $\mathbb{C}^3$ onto $\mathbb{R}^6$, e.g., $(\omega, -1, 0) \mapsto (-\frac{1}{2}, \frac{\sqrt{3}}{2}, -1, 0, 0, 0)$, we obtain 9 two-dimensional planes in $\mathbb{R}^6$.

In this example all principle angles between each pair of planes are in fact equal to $\frac{\pi}{3}$. In particular, the chordal distance is $d_c^2 = \frac{3}{2}$, which can easily be seen to satisfy the simplex bound (cf. (14)). By Theorem 4.3 it now follows that the fusion frame consisting of these planes is tight, and Proposition 4.2 shows that the frame bound equals 3.

Example 5.3 The third example explores the construction of fusion frames in $\mathbb{R}^8$ by employing a similar strategy as in Example 5.2. However, with this example we wish to illustrate the need to be particularly meticulous when generating a fusion frame from minimal vectors of a particular lattice. In fact

by using a similar approach, we will generate a tight fusion frame with equi-dimensional subspaces, but not equi-distance subspaces, although with a very distinct set of chordal distances.

For our analysis, we choose the lattice

$$E_8 = \left\{ (x_1, \ldots, x_8) : \left(x_i \in \mathbb{Z} \ \forall \ 1 \leq i \leq 8 \text{ or } x_i \in \mathbb{Z} + \tfrac{1}{2} \ \forall \ 1 \leq i \leq 8 \right) \right.$$
$$\left. \text{and } \textstyle\sum_{i=1}^8 x_i \in 2\mathbb{Z} \right\},$$

which is again a lattice over the Eisenstein integers $\mathcal{E} = \{a + \omega b : a, b \in \mathbb{Z}\}$, $\omega = \frac{-1+i\sqrt{3}}{2}$. Before studying the minimal vectors in this lattice, we consider the complex root of unity $\omega = \frac{-1+i\sqrt{3}}{2}$ which was employed in the construction of $\mathcal{E}$. We first express ω in quaternions, which gives $\omega = \frac{1}{2}(-1 + i + j + k)$. Next we define a matrix $\mathbf{H}$ by choosing as row vectors the coefficients of ω, $i\omega$, $j\omega$, and $k\omega$, i.e.,

$$\mathbf{H} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix}.$$

Form this, we build an 8×8-matrix by setting

$$\mathbf{\Omega} = \begin{pmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{pmatrix}.$$

Realizing that this matrix satisfies $\mathbf{\Omega}^2 + \mathbf{\Omega} + \mathbf{I} = 0$, we can conclude that scaling a vector $\mathbf{v} \in E_8$ by an Eisenstein integer $a + \omega b$ can be rewritten as

$$(a + \omega b)\mathbf{v} = a\mathbf{v} + b\mathbf{\Omega}\mathbf{v}.$$

Now we are equipped to generate subspaces by minimal vectors, whose norm can be computed to equal 2. The lattice E_8 has 240 minimal vectors, which we assign to planes in the following way. We first consider the four minimal vectors

$$(1, -1, 0, 0, 0, 0, 0, 0), \ (1, 0, -1, 0, 0, 0, 0, 0),$$
$$(1, 0, 0, -1, 0, 0, 0, 0), \ (0, 1, -1, 0, 0, 0, 0, 0)$$

and multiply each of them with

$$\mathbf{I}, \ -\mathbf{I}, \ \mathbf{\Omega}, \ -\mathbf{\Omega}, \ \mathbf{I} + \mathbf{\Omega}, \text{ and } -\mathbf{I} - \mathbf{\Omega}. \tag{17}$$

This procedure generates four sets of six minimal vectors, where each set generates a two-dimensional plane in $\mathbb{R}^8$. Noticing that this construction only

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takes all minimal vectors which are of the form $(x_1, x_2, x_3, x_4, 0, 0, 0, 0)$ into account, we can clearly use the same idea to group all minimal vectors of the form $(0, 0, 0, 0, x_5, x_6, x_7, x_8)$. Summarizing, this construction provides us with 8 two-dimensional planes in $\mathbb{R}^8$ which we denote by $\mathcal{W}_1, \ldots, \mathcal{W}_8$. Next we consider minimal vectors $(x_1, \ldots, x_8)$, which have one coordinate out of x_1, x_2, x_3, x_4 and one coordinate out of x_5, x_6, x_7, x_8 equal to -1 or 1, the others being equal to zero. Again we multiply these vectors by the factors given in (17). We can easily see that this procedure generates another 32 two-dimensional planes in $\mathbb{R}^8$, denoted by $\mathcal{W}_9, \ldots, \mathcal{W}_{40}$.

Although this construction seems similar to the one on Example 5.2, we found it surprising to see that in fact $\{\mathcal{W}_i\}_{i=1}^{40}$ does constitute an equi-dimension tight fusion frame, however the subspaces are not equi-distance. The fusion frame bound can be derived from Proposition 4.2 and equals 10. Most interestingly, the structure of the chordal distances is rather distinct. In fact, it can be computed that the chordal distance between each pair is either $d_c^2 = 2$ – which means that they are orthogonal – or $d_c^2 = \frac{4}{3}$.

6 Conclusions

We considered the linear estimation of a random vector from its noisy projections onto low-dimensional subspaces constituting a fusion frame. We proved that – in the presence of white noise – the MSE in such an estimation is minimal when the fusion frame is tight. We analyzed the effect of subspace erasures on the performance of LMMSE estimators. We proved that maximum robustness against one subspace erasures is achieved when the fusion frame is tight and all subspaces have equal dimensions, where the optimal dimension depends on the SNR. We also proved that equi-distance tight fusion frames are maximally robust against two and more than two subspace erasures. In addition we proved that equi-distance tight fusion frames are in fact optimal Grassmannian packings, and thereby showed that optimal Grassmannian packings are fundamental for signal processing applications where low-dimensional projections are used for robust dimension reduction. We presented a few examples for the construction of equi-distance tight fusion frames and illustrated the interesting and sometimes challenging nature of such constructions.

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